SOME REMARKS ON PRANDTL SYSTEM*

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Abstract. The purpose of this paper is to correct some drawbacks in the proof of the well-known Boundary Layer Theory in Oleinik's book. The Prandtl system for a nonstationary layer arising in an axially symmetric incopressible flow past a solid body is analyzed.

Keywords: Boundary Layer Theory, error, Prandtl system

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1. Prandtl system

At the International Mathematical Congress held in Heidelberg in 1904, Prandtl, in his lecture "Fluid motion with very small friction", suggested a new theory, currently called the *theory of boundary layer*. He showed that the flow about a solid body can be divided into two regions: a very thin layer in the neighborhood of the body (the boundary layer) where viscous friction plays an essential role, and the region outside this layer where friction may be neglected (the outer flow). Thus, for fluids whose viscosity is small, its influence is perceptible only in a very thin region adjacent to the walls of the body in the flow: this region, according to Prandtl, is called the *boundary layer*. This phenomenon is explained by the fact that the fluid sticks to the surface of the solid body and, owing to friction, this adhesion inhibits the motion of the fluid adjacent to the surface of the solid body. In this thin region the velocity of the flow past a body at rest undergoes a sharp increase: from zero at the surface to the values of the velocity in the outer flow, where the fluid may be regarded as frictionless. Thus, for the Navier-Stokes system describing viscous flows, we observe the phenomenon peculiar to many classes of partial differential equations with a small parameter as a coefficient of the highest order derivatives.

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Prandtl derived a system of equations for the first approximation of the flow velocity in the boundary layer. This system served as a basis for the development of the boundary layer theory, which has now become one of the fundamental parts of fluid dynamics. There is a vast literature on theoretical and experimental aspects of that theory. Mathematical methods have an important place in the theory of boundary layer. Mathematical studies of the Prandtl system reveal the nature of the equations governing the flow within the boundary layer and, thereby, provide a description of the laws (in their qualitative and quantitative aspects) underlying the motion of fluids with small viscosity. This approach requires an investigation of such topics as the well-posedness of various boundary value problems and of stability of their solutions with respect to perturbations of the given quantities. Another group of problems deals with the qualitative behavior of the solutions and their asymptotics. Finally, of great importance for applications are the methods for approximate solution of the Prandtl system and subsequent evaluation of the rate of convergence of the approximations to the exact solution.

Among lots of results in boundary theory, the classical and well-known results were obtained by Oleinik, see her book [1]. Oleinik's methods and results are very beautiful. However, when reading [1] thoroughly, we find that there are some errors in the proofs. Though perhaps these errors are not so essential, they nonetheless are not simple errors such as misprints and they actually make reading [1] more difficult. From the point of view of completeness of mathematics, we think it is necessary to correct these errors. It took us much time to do the modification. We only discuss the Prandtl system for a nonstationary layer arising in an axially symmetric incompressible flow past a solid body. We would like to point out that there also exist the same errors in the discussion of the other nonstationary Prandtl systems in [1], and all these errors can be corrected as is shown in our paper. Our proofs follow the outline in [1]; we will point the errors in the form of problems and then give methods of how to solve these problems.

The Prandtl system for a nonstationary layer arising in an axially symmetric incompressible flow past a solid body has the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1.1}$$

$$\frac{\partial (ru)}{\partial x}+\frac{\partial (rv)}{\partial y}=0 \tag{1.2}$$

in a domain $D=\{0<t<T,\ 0<x<X,\ 0<y<\infty\}$, where ν is a positive constant, $U(t,x)$ and $r(x)$ are given functions such that $U(t,0)=0$, $U(t,x)>0$ for $x>0$, $r(0)=0$, $r(x)>0$ for $x>0$.

System (1.1)–(1.2) is considered together with the conditions

$$(1.3)\quad u(0,x,y)=u_0(x,y),\quad u(t,0,y)=0,\quad u(t,x,0)=0,\quad v(t,x,0)=v_0(x),$$

$$(1.4)\qquad u(t,x,y)\to U(t,x)\quad \text{as } y\to\infty.$$

Definition 1. A solution of problem (1.1)–(1.4) is a pair of functions $u(t,x,y)$, $v(t,x,y)$ with the following properties: $u(t,x,y)$ is continuous and bounded in $\overline{D}$; $v(t,x,y)$ is continuous with respect to y in D and bounded for bounded y; the weak derivatives u_t, u_x, u_y, u_{yy}, v_y are bounded measurable functions; equations (1.1)–(1.2) hold for $u, v \in D$, and conditions (1.3)–(1.4) are satisfied.

Introducing the Crocco variables

$$(1.5)\qquad \tau=t,\quad \xi=x,\quad \eta=\frac{u(t,x,y)}{U(t,x)},$$

we obtain the following equation for $w(\tau,\xi,\eta)=u_y(t,x,y)/U(t,x)$:

$$(1.6)\qquad \nu w^2 w_{\eta\eta}-w_\tau-\eta U w_\xi+Aw_\eta+Bw=0$$

in the domain $\Omega=\{0<\tau<T,\ 0<\xi<X,\ 0<\eta<1\}$, where

$$A=(\eta^2-1)U_x+(\eta-1)\frac{U_t}{U},\quad B=\frac{\eta r_x U}{r}-\eta U_x-\frac{U_t}{U}.$$

The initial and the boundary conditions for w have the form

$$(1.7)\qquad w|_{\tau=0}=\frac{u_{0y}}{U}\equiv w_0(\xi,\eta),\quad w|_{\eta=1}=0,\quad (\nu w w_\eta-v_0 w-C)|_{\eta=0}=0,$$

where

$$C=U_x+\frac{U_t}{U}.$$

Solutions of problem (1.6)–(1.7) are understood in the weak sense.

Definition 2. A solution of problem (1.6)–(1.7) is a pair of functions $w(\tau,\xi,\eta)$ with the following properties: w is continuous in $\bar{\Omega}$, the weak derivatives w_τ, w_ξ, w_η are bounded measurable functions, w_η is continuous with respect to η at $\eta=0$ and its weak derivative $w_{\eta\eta}$ is such that $ww_{\eta\eta}$ is bounded in $\bar{\Omega}$; equation (1.6) holds almost everywhere in Ω for w, and conditions (1.3)–(1.4) are satisfied.

Using the line method, we are going to prove, under suitable assumptions on the data, the existence and uniqueness of the solution for problem (1.6)–(1.7) and derive from these results the corresponding existence and uniqueness theorems for problem (1.1)–(1.4).

For any function $f(\tau, \xi, \eta)$, we can use the following notation:

$$f^{m,k}(\eta) \equiv f(mh, kh, \eta), \quad h = \text{const} > 0.$$

Instead of equation (1.6) and conditions (1.7), let us consider the system of ordinary differential equations

$$\begin{aligned}(1.8) \quad \nu(w^{m-1,k} + h)^2 w^{m,k}_{\eta\eta} - \frac{w^{m,k} - w^{m-1,k}}{h} - \eta U^{m,k} \frac{w^{m,k} - w^{m,k-1}}{h} \\ + A^{m,k} w^{m,k}_\eta + B^{m,k} w^{m,k} = 0,\end{aligned}$$

with the conditions

$$\begin{aligned}(1.9) \qquad & w^{m,k}(1) = 0 \\ & \nu w^{m-1,k}(0) w^{m,k}_\eta(0) - v_0^{m,k} w^{m-1,k}(0) + C^{m,k} = 0, \\ & w^{0,k} = w_0^h(kh, \eta)\end{aligned}$$

where $m = 1, \dots, [T/h]$; $k = 0, 1, \dots, [X/h]$ and we take $w_0^h \equiv w_0(\xi, \eta)$ if w_0 has bounded derivatives $w_{0\xi}$, $w_{0\eta}$ and $w_{0\eta\eta}$. If $w_0(\xi, \eta)$ is not so smooth, we take w_0^h for a certain smooth function (to be constructed below) which uniformly converges to w_0 in the domain $0 < \xi < X$, $0 < \eta < 1$ as $h \to 0$.

Finding a solution of (1.8)–(1.9) amounts to consecutively solving linear second order differential equations with given boundary conditions (1.9); first, for $m = 1$, $k = 0, 1, \dots, [X/h]$, then $m = 2$, $k = 0, 1, \dots, [X/h]$, etc.

In what follows K_i, M_i, C_i stand for positive constants independent of h.

Lemma 3. *Assume that A, B, C, v_0 are bounded functions in Ω. Let w_0^h be continuous in $\eta \in [0, 1]$ and such that $K_1(1-\eta) \leqslant w_0^h \leqslant K_2(1-\eta)$. Then problem (1.8)–(1.9) for ordinary differential equations admits a unique solution for $mh \leqslant T_0$ and small enough h, where $T_0 > 0$ is a constant which depends on the data of problem (1.1)–(1.4). The solution w_0^h of problem (1.8)–(1.9) satisfies the estimate*

$$(1.10) \qquad V(mh, \eta) \leqslant w^{m,k}(\eta) \leqslant V_1(mh, \eta),$$

where V and V_1 are continuous functions in $\bar{\Omega}$, positive for $\eta < 1$ and such that $V \equiv K_3(1-\eta)$, $V_1 \equiv K_4(1-\eta)$ in a neighborhood of $\eta = 1$.

For m, k fixed, the linear second order equation (1.8) with the unknown function $w^{m,k}$ and boundary conditions (1.9) admits a solution $w^{m,k}$, if $w^{m-1,k}(0) \neq 0$ and $w^{m-1,k}(\eta) \geqslant 0$, $|B^{m,k}| < h^{-1}$. The existence of this solution follows from its uniqueness which, in turn, can be established on the basis of the maximum principle and the fact that this problem can be reduced, with help of the Green function, to a Fredholm integral equation of the second kind.

Indeed, let $Q^{m,k}$ be the difference of two solutions $w^{m,k}$ of problem (1.8)–(1.9). Then $Q^{m,k}$ can attain neither a positive maximum nor a negative minimum at $\eta = 0$, since otherwise $Q^{m,k}_\eta(0) \neq 0$ (see [2, Lemma 3.4]), whereas the boundary condition (1.9) implies that $Q^{m,k}_\eta(0) = 0$. We also have $Q^{m,k}(1) = 0$, and at the interior points of [0,1] this difference can neither attain a positive maximum nor a negative minimum, since $\max |B^{m,k}| < h^{-1}$. Consequently, under our assumptions, problem (1.7)–(1.9) cannot have more than one solution. Therefore, we shall a *fortiori* establish solvability of problem (1.8)–(1.9) for m and j such that the solutions w of problem (1.8)–(1.9) admit the following a priori estimate:

$$w^{m-1,k}(\eta) \geqslant V((m-1)h, \eta).$$

In order to prove the a priori estimate (1.10) for $\tau = mh$, it suffices to show that there exist functions V and V_1 with the properties specified in Lemma 3 and such that

$$\begin{aligned}(1.11)\qquad L_m(V) \equiv \nu(w^{m-1,k} + h)^2 V^{m,k}_{\eta\eta} &- \frac{V^{m,k} - V^{m-1,k}}{h} \\ - \eta U^{m,k} \frac{V^{m,k} - V^{m,k-1}}{h} &+ A^{m,k} V^{m,k}_\eta + B^{m,k} V^{m,k} \geqslant 0,\end{aligned}$$

$$(1.12)\qquad \lambda_m(V) \equiv \nu w^{m-1,k}(0) V^{m,k}_\eta(0) - v^{m,k}_0 w^{m-1,k}(0) - C^{m,k} > 0,$$

$$(1.13)\qquad L_m(V_1) \leqslant 0, \quad \lambda_m(V_1) < 0, \quad k = 0, 1, \ldots, [X/h],$$

under the assumption that

$$(1.14)\qquad V((m-1)h, \eta) \leqslant w^{m-1,k}(\eta) \leqslant V_1((m-1)h, \eta).$$

Then inequality (1.10) can be proved by induction with respect to m. Indeed, consider the function $q^{m,k} = V(mh, \eta) - w^{m,k}$, where $w^{m,k}$ is the solution of problem (1.8)–(1.9). We have

$$L_m(q) \geqslant 0, \quad \lambda_m(V) - \lambda_m(w) \equiv \upsilon w^{m-1,k}(0) q^{m,k}_\eta(0) > 0.$$

Moreover, by assumption we have $q^{m',k} \leqslant 0$ for $m' \leqslant m-1$ and $q^{m,k} = 0$ for $\eta = 1$. Let us show that $q^{m,k} \leqslant 0$. To this end, we introduce new functions by $q^{m,k} = \mathrm{e}^{\alpha m h} S^{m,k}$, where $\alpha > 0$ is a constant to be chosen below. Then

$$\begin{aligned} L_m(q) = \mathrm{e}^{\alpha m h}\Big[&\nu(w^{m-1,k}+h)^2 S^{m,k}_{\eta\eta} - \eta U^{m,k}\frac{S^{m,k}-S^{m,k-1}}{h} \\ &+ A^{m,k}S^{m,k}_{\eta} + B^{m,k}S^{m,k} - \frac{(1-\mathrm{e}^{-\alpha h})S^{m,k}}{h} \\ &- \mathrm{e}^{-\alpha h}\frac{S^{m,k}-S^{m-1,k}}{h}\Big] \geqslant 0, \end{aligned} \tag{1.15}$$

$$\lambda_m(V) - \lambda_m(w) = \mathrm{e}^{\alpha m h}\upsilon w^{m-1,k}(0) S^{m,k}_{\eta}(0) > 0. \tag{1.16}$$

It follows that $S^{m,k} \leqslant 0$. Indeed, $S^{m,k}$ cannot assume the maximum positive value at $\eta = 0$ since $S^{m,k}_{\eta}(0) > 0$. Moreover, $S^{m,k} = 0$ for $\eta = 1$. If $S^{m,k}$ attains its maximum positive value at an interior point of the interval $0 \leqslant \eta \leqslant 1$, then at this point, [1] claimed that

$$\begin{gathered} S^{m,k}_{\eta\eta} \leqslant 0, \quad S^{m,k}_{\eta} = 0, \quad \frac{S^{m,k}-S^{m-1,k}}{h} \geqslant 0, \\ \eta U^{m,k}\frac{S^{m,k}-S^{m,k-1}}{h} \geqslant 0, \quad \Big[B^{m,k} - \frac{(1-\mathrm{e}^{-\alpha h})}{h}\Big]S^{m,k} < 0 \end{gathered} \tag{1.17}$$

provided that the constant α is large enough and h is sufficiently small, so that $1-\mathrm{e}^{-\alpha h} \geqslant \frac{1}{2}$, and these relations are incompatible with (1.16).

P r o b l e m 4. Why $(S^{m,k} - S^{m,k-1})/h \geqslant 0$ at the maximum point of $S^{m,k}$? Actually only if $S^{m,k-1} \leqslant 0$ or $S^{m,k} - S^{m,k-1} \geqslant 0$, then $(S^{m,k} - S^{m,k-1})/h \geqslant 0$. But generally, we cannot deduce that $S^{m,k-1} \leqslant 0$ or $S^{m,k} - S^{m,k-1} \geqslant 0$.

So in order to show that $S^{m,k}$ cannot attain its maximum value in the interior of $\eta \in [0,1]$, beside discussing the case (1.17), at the maximum point of $S^{m,k}$, $(S^{m,k} - S^{m,k-1})/h \geqslant 0$, we also need to discuss the case of $S^{m,k-1} \geqslant 0$, $S^{m,k} - S^{m,k-1} \leqslant 0$. Let us set

$$S^{m,k}_1 = \mathrm{e}^{-\beta k h} S^{m,k}$$

where β is a constant chosen below. Then

$$\begin{aligned} \mathrm{e}^{\alpha m h}\Big[&\nu(w^{m-1,k}+h)^2 \mathrm{e}^{\beta k h} S^{m,k}_{1\eta\eta} - \eta U^{m,k}\frac{\mathrm{e}^{\beta k h}S^{m,k}_1 - \mathrm{e}^{\beta(k-1)h}S^{m,k-1}_1}{h} \\ &+ A^{m,k}\mathrm{e}^{\beta k h}S^{m,k}_{1\eta} + B^{m,k}\mathrm{e}^{\beta k h}S^{m,k}_1 - \frac{(1-\mathrm{e}^{-\alpha h})\mathrm{e}^{\beta k h}S^{m,k}_1}{h} \\ &- \mathrm{e}^{-\alpha h}\mathrm{e}^{\beta k h}\frac{S^{m,k}_1 - S^{m-1,k}_1}{h}\Big] \geqslant 0. \end{aligned} \tag{1.18}$$

Clearly at the maximum point

$$
\begin{aligned}
(1.19)\qquad & -\eta U^{m,k}\frac{\mathrm{e}^{\beta kh}S_1^{m,k}-\mathrm{e}^{\beta(k-1)h}S_1^{m,k-1}}{h}\\
&=-\frac{\eta U^{m,k}}{h}\mathrm{e}^{\beta(k-1)h}(S_1^{m,k}-S_1^{m,k-1})+\frac{\eta U^{m,k}}{h}(\mathrm{e}^{\beta(k-1)h}-\mathrm{e}^{\beta kh})S_1^{m,k}\\
&\geqslant \frac{\eta U^{m,k}}{h}(\mathrm{e}^{\beta(k-1)h}-\mathrm{e}^{\beta kh})S_1^{m,k},
\end{aligned}
$$

if we choose β a negative constant and $-\beta$ is large enough. Now,

$$
\begin{aligned}
&\mathrm{e}^{\beta kh}\Big(B^{m,k}-\frac{1-\mathrm{e}^{-\alpha h}}{h}-\frac{\mathrm{e}^{-\alpha h}}{h}\Big)+\frac{\eta U^{m,k}}{h}(\mathrm{e}^{\beta(k-1)h}-\mathrm{e}^{\beta kh})\\
&=\mathrm{e}^{\beta kh}\Big[B^{m,k}-\frac{\mathrm{e}^{-\alpha h}}{h}+\frac{\eta U^{m,k}}{h}(\mathrm{e}^{-\beta h}-1)\Big]\leqslant 0,
\end{aligned}
$$

if we choose $\alpha=\alpha(\beta)$ a negative constant and $-\alpha$ is large enough. By (1.18), (1.19), we know that this is also impossible. The above discussion means that $S^{m,k}$ cannot attain its maximum positive value at an interior point of the interval $0\leqslant\eta\leqslant 1$, either $(S^{m,k}-S^{m,k-1})/h\geqslant 0$ or $S^{m,k-1}\geqslant 0$, $S^{m,k}-S^{m,k-1}\leqslant 0$.

Therefore

$$q^{m,k}=\mathrm{e}^{\alpha mh}S^{m,k}\leqslant 0,\quad V(mh,\eta)\leqslant w^{m,k}.$$

In a similar way we can show that (1.13)–(1.14) implies $w^{m,k}\leqslant V_1(mh,\eta)$. For the construction of V, V_1 one can refer to [1], we omit details here.

2. Oleinik's line method

In what follows, we take as $w_0^h(\xi,\eta)$ the function $w_0(\xi,\eta)$ if $w_{0\eta\eta}(\xi,\eta)$ is bounded in Ω; otherwise, we let $w_0^h(\xi,\eta)$ be a function coinciding with w_0 for $\eta\leqslant\frac12$, equal to $w_0(\xi,\eta-h)-w_0(\xi,1-h)$ for $\frac12+h\leqslant\eta<1$ and defined on the interval $\frac12\leqslant\eta\leqslant\frac12+h$ in such a way that for $\frac14\leqslant\eta\leqslant\frac34$ it has uniformly (in h) bounded derivatives which are known to be bounded for w_0.

Lemma 5. *Assume that the conditions of Lemma* 3 *are fulfilled and the functions* A, B, C, v_0, w_0 *have bounded first order derivatives,* $|w_{0\xi}|\leqslant K_5(1-\eta)$, $w_0(\xi,1)=0$, $w_0w_{0\eta\eta}$ *is bounded in* Ω*, and the following compatibility condition is satisfied:*

$$\nu w_0w_{0\eta}-v_0w_0+C=0\quad\text{for } \tau=0,\ \eta=0. \tag{2.1}$$

Then

$$w_\eta^{m,k},\quad \frac{w^{m,k}-w^{m,k-1}}{h},\quad \frac{w^{m,k}-w^{m-1,k}}{h},\quad (1-\eta+h)w_{\eta\eta}^{m,k}, \tag{2.2}$$

are bounded in Ω for $mh \leqslant T_1$ and $h \leqslant h_0$ uniformly with respect to h. The positive constants T_1 and h_0 are determined by the data of problem (1.1)–(1.4)*;* $T_1 \leqslant T_0$*.*

P r o o f. Let us introduce a new unknown function $W^{m,k} = w^{m,k}\mathrm{e}^{\alpha\eta}$ in problem (1.8)–(1.9), where α is a positive constant which does not depend on h and will be chosen later. We have

$$\begin{aligned}(2.3)\qquad \nu(w^{m-1,k}+h)^2 W^{m,k}_{\eta\eta} - \frac{W^{m,k}-W^{m-1,k}}{h} - \eta U^{m,k}\frac{W^{m,k}-W^{m,k-1}}{h}\\ + \tilde{A}^{m,k}W^{m,k}_{n} + \tilde{B}^{m,k}W^{m,k} = 0,\end{aligned}$$

with the conditions

$$(2.4)\quad \nu W^{m-1,k}(0)W^{m,k}_{\eta}(0) - \alpha\upsilon W^{m-1,k}(0)W^{m,k}(0) - v^{m,k}_0 w^{m-1,k}(0) + C^{m,k} = 0$$

where

$$\tilde{A}^{m,k} = A^{m,k} - 2\alpha\nu(w^{m-1,k}+h)^2, \quad \tilde{B}^{m,k} = B^{m,k} - \alpha A^{m,k} + \alpha^2\upsilon(w^{m-1,k}+h)^2.$$

Consider the function $\Phi^{m,k}(\eta)$ defined for $m \geqslant 1$, $k \geqslant 1$ by

$$\Phi^{m,k}(\eta) = (W^{m,k}_{\eta})^2 + \Big(\frac{W^{m,k}-W^{m-1,k}}{h}\Big)^2 + \Big(\frac{W^{m,k}-W^{m,k-1}}{h}\Big)^2 + K_6\eta + 1$$

and for $m \geqslant 1$, $k = 0$ by

$$(2.5)\qquad \Phi^{m,k}(\eta) = (W^{m,k}_{\eta})^2 + \Big(\frac{W^{m,k}-W^{m-1,k}}{h}\Big)^2 + K_6\eta + 1.$$

The constant $K_6 > 0$ will be chosen below. Let us define the function $\Phi^{m,k}(\eta)$ with $m = 0$. For this purpose, we introduce functions $W^{-1,k}$ by

$$(2.6)\quad \frac{W^{0,k}-W^{-1,k}}{h} = \nu(w^{0,k}+h)^2 - \eta U^{0,k}\frac{W^{0,k}-W^{0,k-1}}{h} + \tilde{A}^{0,k}W^{0,k}_{\eta} + \tilde{B}^{0,k},$$

where

$$\begin{aligned}\tilde{A}^{0,k} &= A^{0,k} - 2\alpha\nu(w^{0,k}\mathrm{e}^{-\alpha\eta}+h)^2,\\ \tilde{B}^{0,k} &= B^{0,k} - \alpha A^{0,k} + \alpha^2\upsilon(w^{0,k}\mathrm{e}^{-\alpha\eta}+h)^2.\end{aligned}$$

Then we define the function $\Phi^{0,k}$ for $k \geqslant 1$ and $k = 0$ by (2.4)–(2.5). By [1], we have

Claim 1.

$$(2.7)\qquad |\Phi^{0,k}| \leqslant K_{15},$$

Claim 2. *When* $mh \leqslant T_0$,

$$\Phi_\eta^{m,k}(0) \geqslant \frac{\alpha}{2}\Phi^{m,k}(0) - \frac{\alpha}{4}\Phi^{m-1,k}(0), \qquad m = 2,3,\dots,[T_0/h],\ \ k = 0,1,2,\dots,[X/h], \tag{2.8}$$

$$\Phi_\eta^{1,k}(0) \geqslant \frac{\alpha}{2}\Phi^{1,k}(0), \quad k = 0,1,2,\dots,[X/h]. \tag{2.9}$$

We introduce functions

$$r^{m,k} = h^{-1}(W^{m,k} - W^{m,k-1}), \quad \varrho^{m,k} = h^{-1}(W^{m,k} - W^{m-1,k}).$$

Let us write down the differential equations which hold for $\Phi^{m,k}$ on the interval $0 \leqslant \eta < 1$. To this end, we differentiate equation (2.3) in η and multiply the result by $2W_\eta^{m,k}$; then we subtract from equation (2.3) for $W^{m,k}$ equation (2.3) for $W^{m-1,k}$ and multiply the result by $2\varrho^{m,k}/h$; from (2.3) for $W^{m,k}$ we subtract (2.3) for $W^{m,k-1}$ and multiply the result by $2r^{m,k}/h$. Taking the sum of the three equations just obtained we get the equation for $\Phi^{m,k}$, $m = 1,2,3,\dots,[T_0/h]$, $k = 0,1,2,\dots,[X/h]$. In detail,

$$\begin{aligned}\nu(w^{m-1,k}+h)^2 2W_\eta^{m,k}W_{\eta\eta\eta}^{m,k} - 2W_\eta^{m,k}\varrho_\eta^{m,k} - 2W_\eta^{m,k}\eta U^{m,k}r_\eta^{m,k}\\ +2W_\eta^{m,k}\tilde{A}^{m,k}W_{\eta\eta}^{m,k} + 2\tilde{B}^{m,k}(W_\eta^{m,k})^2 + 4\upsilon(w^{m-1,k}+h)w_\eta^{m-1,k}W_\eta^{m,k}W_{\eta\eta}^{m,k}\\ -\eta U^{m,k}2W_\eta^{m,k}r^{m,k} + 2(W_\eta^{m,k})^2\tilde{A}_\eta^{m,k} + 2W_\eta^{m,k}\tilde{B}_\eta^{m,k}W^{m,k} = 0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\nu(w^{m-1,k}+h)^2 2\varrho^{m,k}\varrho_{\eta\eta}^{m,k} - 2\varrho^{m,k}\frac{\varrho^{m,k}-\varrho^{m-1,k}}{h}\\ -2\varrho^{m,k}\eta U^{m,k}\frac{r^{m,k}-r^{m-1,k}}{h} + 2\varrho^{m,k}\tilde{A}^{m,k}\varrho_\eta^{m,k} + 2(\varrho^{m,k})^2\tilde{B}^{m,k}\\ +\frac{2\varrho^{m,k}}{h}\big(\upsilon[(w^{m-1,k}+h)^2 - (w^{m-2,k}+h)^2]W_{\eta\eta}^{m-1,k} - \eta r^{m-1,k}(U^{m,k}-U^{m-1,k})\big)\\ +\frac{2\varrho^{m,k}}{h}\big(W_\eta^{m-1,k}(\tilde{A}^{m,k}-\tilde{A}^{m-1,k}) + W^{m-1,k}(\tilde{B}_\eta^{m,k}-\tilde{B}_\eta^{m-1,k})\big) = 0,\end{aligned} \tag{2.11}$$

$$\begin{aligned}\nu(w^{m-1,k}+h)^2 2r^{m,k}r_{\eta\eta}^{m,k} - 2r^{m,k}\frac{\varrho^{m,k}-\varrho^{m,k-1}}{h}\\ -2r^{m,k}\eta U^{m,k}\frac{r^{m,k}-r^{m,k-1}}{h} + 2r^{m,k}\tilde{A}^{m,k}r_\eta^{m,k} + 2(r^{m,k})^2\tilde{B}^{m,k}\\ +\frac{2r^{m,k}}{h}\big(\upsilon[(w^{m-1,k}+h)^2 - (w^{m-2,k}+h)^2]W_{\eta\eta}^{m-1,k} - \eta r^{m,k-1}(U^{m,k}-U^{m,k-1})\big)\\ +\frac{2r^{m,k}}{h}\big(W_\eta^{m-1,k}(\tilde{A}^{m,k}-\tilde{A}^{m,k-1}) + W^{m,k-1}(\tilde{B}_\eta^{m,k}-\tilde{B}_\eta^{m,k-1})\big) = 0;\end{aligned} \tag{2.12}$$

$$\Phi_\eta^{m,k} = 2W_\eta^{m,k}W_{\eta\eta}^{m,k} + 2\varrho^{m,k}\varrho_\eta^{m,k} + 2r^{m,k}r_\eta^{m,k} + K_6,$$

$$\Phi_{\eta\eta}^{m,k} = 2(W_{\eta\eta}^{m,k})^2 + 2W_\eta^{m,k}W_{\eta\eta\eta}^{m,k} + 2(\varrho_\eta^{m,k})^2 + 2\varrho^{m,k}\varrho_{\eta\eta}^{m,k} + 2(r_\eta^{m,k})^2 + 2r^{m,k}r_{\eta\eta}^{m,k},$$

$$
\begin{aligned}
(2.13)\qquad & \nu(w^{m-1,k}+h)^2\Phi^{m,k}_{\eta\eta}-\frac{\Phi^{m,k}-\Phi^{m-1,k}}{h}\\
& -\eta U^{m,k}\frac{\Phi^{m,k}-\Phi^{m,k-1}}{h}+\tilde{A}^{m,k}\Phi^{m,k}_{n}+\tilde{B}^{m,k}\Phi^{m,k}\\
& =2\nu(w^{m-1,k}+h)^2[(W^{m,k}_{\eta\eta})^2+(\varrho^{m,k}_{\eta})^2+(r^{m,k}_{\eta})^2]+2W^{m,k}_{\eta}\varrho^{m,k}_{\eta}\\
& +2\eta U^{m,k}W^{m,k}_{\eta}r^{m,k}_{\eta}-2\tilde{A}^{m,k}W^{m,k}_{\eta}W^{m,k}_{\eta\eta}\\
& -2\tilde{B}^{m,k}(W^{m,k}_{\eta})^2-\upsilon a^{m-1,k}_{\eta}W^{m,k}_{\eta}W^{m,k}_{\eta\eta}+2\eta U^{m,k}W^{m,k}_{\eta}r^{m,k}\\
& -2\tilde{A}^{m,k}(W^{m,k}_{\eta})^2-2\tilde{B}^{m,k}W^{m,k}_{\eta}w^{m,k}\\
& +\frac{2\varrho^{m,k}(\varrho^{m,k}-\varrho^{m-1,k})}{h}+\frac{2\eta U^{m,k}}{h}\varrho^{m,k}(r^{m,k}-r^{m-1,k})\\
& -2\varrho^{m,k}\varrho^{m,k}_{\eta}\tilde{A}^{m,k}-2(\varrho^{m,k})^2\tilde{B}^{m,k}\\
& -\frac{2\upsilon}{h}\varrho^{m,k}(a^{m-1,k}-a^{m-2,k})W^{m-1,k}_{\eta\eta}+\frac{2}{h}\varrho^{m,k}\eta r^{m-1,k}(U^{m,k}-U^{m-1,k})\\
& -\frac{2}{h}\varrho^{m,k}(\tilde{A}^{m,k}-\tilde{A}^{m-1,k})W^{m-1,k}_{\eta}\\
& -\frac{2}{h}\varrho^{m,k}(\tilde{B}^{m,k}-\tilde{B}^{m-1,k})w^{m-1,k}+\frac{2r^{m,k}}{h}(\varrho^{m,k}-\varrho^{m,k-1})\\
& +\frac{2\eta U^{m,k}}{h}r^{m,k}(r^{m,k}-r^{m,k-1})-2r^{m,k}\tilde{A}^{m,k}r^{m,k-1}_{\eta}-2\tilde{B}^{m,k}(r^{m,k})^2\\
& -\frac{2r^{m,k}}{h}\big((a^{m-1,k}-a^{m-1,k-1})W^{m,k-1}_{\eta\eta}-\eta r^{m,k-1}(U^{m,k}-U^{m,k-1})\big)\\
& -\frac{2r^{m,k}}{h}W^{m,k-1}_{\eta}(\tilde{A}^{m,k}-\tilde{A}^{m,k-1})-\frac{2r^{m,k}}{h}W^{m,k-1}(\tilde{B}^{m,k}-\tilde{B}^{m,k-1})\\
& -\frac{1}{h}[(W^{m,k}_{\eta})^2-(W^{m-1,k}_{\eta})^2+(\varrho^{m,k})^2-(\varrho^{m-1,k})^2+(r^{m,k})^2-(r^{m-1,k})^2]\\
& -\frac{\eta U^{m,k}}{h}[(W^{m,k}_{\eta})^2-(W^{m,k-1}_{\eta})^2+(\varrho^{m,k})^2-(\varrho^{m,k-1})^2+(r^{m,k})^2-(r^{m,k-1})^2]\\
& +\tilde{A}^{m,k}[2W^{m,k}_{\eta}W^{m,k}_{\eta\eta}+2\varrho^{m,k}\varrho^{m,k}_{\eta}+2r^{m,k}r^{m,k}_{\eta}+K_6]\\
& +\tilde{B}^{m,k}[(W^{m,k}_{\eta})^2+(\varrho^{m,k})^2+(r^{m,k})^2],
\end{aligned}
$$

where $a^{m,k}$ stands for $(w^{m,k}+h)^2$.

We find equations for $\Phi^{m,k}(\eta)$ with $k=0$, $m\geqslant 1$ by taking the sum of only the first and the second of these equations. In order to derive the equation for $\Phi^{m,k}(\eta)$ with $m=1$, we utilize the relation (2.6) which determines the values of $W^{-1,k}$. By the above discussion, [1] declared: For $\Phi^{m,k}(\eta)$ with $k\geqslant 1$, $m\geqslant 1$,

$$
\begin{aligned}
(2.14)\quad & \nu(w^{m-1,k}+h)^2\Phi^{m,k}_{\eta\eta}-\frac{\Phi^{m,k}-\Phi^{m-1,k}}{h}-\eta U^{m,k}\frac{\Phi^{m,k}-\Phi^{m,k-1}}{h}\\
& +\tilde{A}^{m,k}\Phi^{m,k}_{n}+\tilde{B}^{m,k}\Phi^{m,k}+N^{m,k}_{1}-N^{m,k}_{2}=0,
\end{aligned}
$$

where $N_2^{m,k}$ is the sum of nonnegative terms:

$$
\begin{aligned}
N_2^{m,k} &= 2\upsilon a^{m-1,k}(W_{\eta\eta}^{m,k})^2 + \frac{1}{h}(\varrho_\eta^{m,k})^2 + \frac{\eta U^{m,k}}{h}(r_\eta^{m,k})^2 \\
&\quad + 2\nu a^{m-1,k}(\varrho_\eta^{m,k})^2 + \frac{1}{h}(\varrho^{m,k} - \varrho^{m-1,k})^2 \\
&\quad + \frac{\eta U^{m,k}}{h}(r^{m,k} - r^{m-1,k})^2 + 2\upsilon a^{m-1,k}(r_\eta^{m,k})^2 \\
&\quad + \frac{1}{h}(\varrho^{m,k} - \varrho^{m,k-1})^2 + \frac{\eta U^{m,k}}{h}(r^{m,k} - r^{m,k-1})^2,
\end{aligned} \tag{2.15}
$$

and $N_1^{m,k}$ is a linear function whose coefficients are uniformly bounded in h and can be expressed in terms of the following quantities:

$$
\begin{gathered}
a_\eta^{m-1,k} W_{\eta\eta}^{m,k} W_\eta^{m,k}, \quad W_\eta^{m,k} r^{m,k}, \quad W_\eta^{m,k} W_\eta^{m-1,k}, \\
(W_\eta^{m,k})^2, \quad W_\eta^{m,k}, \quad (W_\eta^{m,k})^2 W_\eta^{m-1,k}, \quad \varrho^{m,k} r^{m-1,k}, \\
\frac{1}{h}(a^{m-1,k} - a^{m-2,k})\varrho^{m,k} W_{\eta\eta}^{m-1,k}, \quad W_\eta^{m-1,k}\varrho^{m,k}\varrho^{m-1,k}, \\
W_\eta^{m-1,k}\varrho^{m,k}, \quad \varrho^{m,k}\varrho^{m-1,k}, \quad \varrho^{m,k}, \\
\frac{1}{h} W_{\eta\eta}^{m,k-1} r^{m,k}(a^{m-1,k} - a^{m-1,k-1}), \quad r^{m,k} r^{m,k-1}, \quad W_\eta^{m-1,k} r^{m,k} r^{m-1,k}, \\
r^{m,k} r^{m-1,k}, \quad r^{m,k}, \quad W_\eta^{m,k-1} r^{m,k}.
\end{gathered}
$$

P r o b l e m 6. It is impossible to derive $(\varrho_\eta^{m,k})^2/h + \eta U^{m,k}(r_\eta^{m,k})^2/h$ from (2.15).

Taking into account the calculations of (2.10)–(2.14), we believe that (2.15) should be modified to the expression

$$
\begin{aligned}
N_2^{m,k} &= 2\nu a^{m-1,k}(W_{\eta\eta}^{m,k})^2 + \frac{1}{h}(w_\eta^{m,k})^2 + \frac{\eta U^{m,k}}{h}(w_\eta^{m,k})^2 \\
&\quad + 2\nu a^{m-1,k}(\varrho_\eta^{m,k})^2 + \frac{1}{h}(\varrho^{m,k} - \varrho^{m-1,k})^2 \\
&\quad + \frac{\eta U^{m,k}}{h}(r^{m,k} - r^{m-1,k})^2 + 2\nu a^{m-1,k}(r_\eta^{m,k})^2 \\
&\quad + \frac{1}{h}(\varrho^{m,k} - \varrho^{m,k-1})^2 + \frac{\eta U^{m,k}}{h}(r^{m,k} - r^{m,k-1})^2.
\end{aligned} \tag{2.16}
$$

Using the inequality

$$2ab \leqslant \varepsilon a^2 + \frac{b^2}{\varepsilon}, \quad \varepsilon > 0$$

to estimate the terms that make up $N_1^{m,k}$, we obtain from (2.14) (for details one can refer to [1])

$$\begin{aligned}(2.17)\qquad & \nu(w^{m-1,k}+h)^2\Phi_{\eta\eta}^{m,k} - \frac{\Phi^{m,k}-\Phi^{m-1,k}}{h} - \eta U^{m,k}\frac{\Phi^{m,k}-\Phi^{m,k-1}}{h} \\ & + \tilde{A}^{m,k}\Phi_n^{m,k} + 2\tilde{B}^{m,k}\Phi^{m,k} + \tilde{C}^{m,k}\Phi^{m,k} \geqslant 0,\end{aligned}$$

where $\tilde{C}^{m,k}$ depends on

$$w_\eta^{m-1,k},\quad \varrho^{m-1,k},\quad r^{m-1,k},\quad (w^{m-1,k}+w^{m-2,k}+2h)w_{\eta\eta}^{m-1,k},$$
$$(w^{m-1,k}+w^{m-1,k-1}+2h)w_{\eta\eta}^{m,k-1},\quad r^{m,k-1},\quad w_\eta^{m,k-1}.$$

It is easy to see that for $k=1$ the coefficient $\tilde{C}^{m,k}$ does not depend on $r^{m-1,k}$, since $U^{m,0}=0$. The inequality (2.17) for $\Phi^{m,k}$ with $k=0$ is obtained in exactly the same manner as for $k \geqslant 1$. Obviously, in this case the coefficient $\tilde{C}^{m,k}$ depends only on

$$w_\eta^{m-1,k},\quad \varrho^{m-1,k},\quad (w^{m-1,k}+w^{m-2,k}+2h)w_{\eta\eta}^{m-1,k}.$$

Now consider the functions

$$(2.18)\qquad Y^{m,k}(\eta) = (r^{m,k})^2 + (\varrho^{m,k})^2 + f(\eta) \quad \text{for } k \geqslant 1,\ m \geqslant 0;$$

$$(2.19)\qquad Y^{m,k}(\eta) = (\varrho^{m,k})^2 + f(\eta) \quad \text{for } k = 0,\ m \geqslant 0,$$

where $f(\eta) = \kappa(\beta\eta)\kappa_1^2(\eta)$ (for details, one can refer to [1, p. 157 and p. 163]), β is a positive constant. Just as we have proved inequalities (2.8), (2.9), (2.17), we are able to prove the inequalities

$$(2.20)\qquad Y_\eta^{m,k}(0) \geqslant \frac{\alpha}{2}Y^{m,k}(0) - \frac{\alpha}{4}Y^{m-1,k}(0), \quad m>1,\ k \geqslant m,$$

$$Y_\eta^{m,k}(0) \geqslant \frac{\alpha}{2}Y^{m,k}(0), \quad m=1,$$

$$\begin{aligned}& \nu(w^{m-1,k}+h)^2 Y_{\eta\eta}^{m,k} - \frac{Y^{m,k}-Y^{m-1,k}}{h} - \eta U^{m,k}\frac{Y^{m,k}-Y^{m,k-1}}{h} + \tilde{A}^{m,k}Y_n^{m,k} \\ & + 2\tilde{B}^{m,k}Y^{m,k} + Q_1^{m,k}Y^{m,k} + Q_2^{m,k} + Q_3 \geqslant 0,\end{aligned}$$

where $Q_1^{m,k} \geqslant 0$. For the definitions of Q_1, Q_2, Q_3, one can refer to [1] for details.

Let us show by induction that

$$(2.21)\qquad Y^{m,k} \leqslant M_2(1-\eta+h)^2, \quad \Phi^{m,k} \leqslant M_1.$$

For $mh \leqslant T_1$ and some $T_1 \leqslant T_0$, the constants T_1 and M_i are independent of h. To show this, assume that for $m < m'$ the inequalities (2.21) hold with constants M_1,

M_2 specified below. Let us show that if $mh \leqslant T_1$ the same inequalities are valid for $m = m'$. Note that under the induction assumptions, we can claim that for $m < m'$ or $m = m'$, $k < k'$, the following inequalities hold:

$$\begin{aligned}
|W_{\eta\eta}^{m-1,k}&(w^{m-1,k} + w^{m-2,k} + 2h)| \\
&\leqslant K_{25}(1-\eta+h)|W_{\eta\eta}^{m-1,k}| \\
&\leqslant K_{26}\upsilon(w^{m-2,k} + h)|W_{\eta\eta}^{m-1,k}| \\
&= K_{26}|\varrho^{m-1,k} + \eta U^{m,k} r^{m-1,k} - \tilde{A}^{m-1,k} W_{\eta}^{m-1,k} \\
&\quad - \tilde{B}^{m-1,k} W^{m-1,k}|(w^{m-2,k} + h)^{-1} \leqslant K_{27}.
\end{aligned}$$

In exactly the same manner we find that

$$|W_{\eta\eta}^{m,k-1}(w^{m-1,k} + w^{m-1,k-1} + 2h)| \leqslant K_{28}.$$

The constants K_{27} and K_{28} depend on M_1 and M_2. Therefore, if the inequalities (2.21) hold for $m < m'$ and for $m = m'$, $k < k'$, then it can be seen that in (2.17) and (2.20) we have

$$\begin{aligned}
|\tilde{C}^{m,k}| &\leqslant K_{29}(M_1, M_2), \\
|Q_1^{m,k}| &\leqslant K_{30}(M_1, M_2), \\
|Q_2^{m,k}| &\leqslant K_{31}(1-\eta+h)^2.
\end{aligned} \tag{2.22}$$

Let us pass to new functions in (2.17) and (2.10) by

$$\Phi^{m,k} = \tilde{\Phi}^{m,k} \mathrm{e}^{\gamma m h}, \quad Y^{m,k} = \tilde{Y}^{m,k} \mathrm{e}^{\gamma m h}. \tag{2.23}$$

The constant $\gamma(M_1, M_2)$ will be chosen later. For $1 \leqslant m \leqslant m'$ and $m = m'$, we have

$$\begin{aligned}
\nu a^{m-1,k} \tilde{\Phi}_{\eta\eta}^{m,k} &- \mathrm{e}^{-\gamma h} \frac{\tilde{\Phi}^{m,k} - \tilde{\Phi}^{m-1,k}}{h} - \eta U^{m,k} \frac{\tilde{\Phi}^{m,k} - \tilde{\Phi}^{m,k-1}}{h} \\
&+ \tilde{A}^{m,k} \tilde{\Phi}_{n}^{m,k} + (2\tilde{B}^{m,k} + \tilde{C}^{m,k} - \gamma \mathrm{e}^{-\gamma h'}) \tilde{\Phi}^{m,k} \geqslant 0
\end{aligned} \tag{2.24}$$

for $0 < h' < h$, and also

$$\begin{aligned}
\nu a^{m-1,k} \tilde{Y}_{\eta\eta}^{m,k} &- \mathrm{e}^{-\gamma h} \frac{\tilde{Y}^{m,k} - \tilde{Y}^{m-1,k}}{h} - \eta U^{m,k} \frac{\tilde{Y}^{m,k} - \tilde{Y}^{m,k-1}}{h} \\
&+ \tilde{A}^{m,k} \tilde{Y}_{\eta}^{m,k} + (2\tilde{B}^{m,k} + Q_1^{m,k} - \gamma \mathrm{e}^{-\gamma h'}) \tilde{Y}^{m,k} \\
&+ K_{32}(M_1, M_2)(1-\eta+h)^2 \geqslant 0.
\end{aligned} \tag{2.25}$$

Let us choose $\gamma(M_1, M_2)$ such that for small enough h the following inequalities are valid:

$$2\tilde{B}^{m,k} + \tilde{C}^{m,k} - \gamma e^{-\gamma h'} < 0, \quad 2\tilde{B}^{m,k} + \tilde{C}^{m,k} - \gamma e^{-\gamma h'} < -K_{33}(M_1, M_2), \tag{2.26}$$

where

$$K_{33} = \frac{2K_{32}}{M_2} + K_{34}, \quad K_{34}(1-\eta+h)^2 \geqslant |2\nu a^{m-1,k} - 2\tilde{A}^{m,k}(1-\eta+h)|.$$

Consider the point at which $\tilde{\Phi}^{m,k}$, for $0 < \eta < 1$, $m < m'$ or $m = m'$, $k \leqslant k'$, attains its largest value. In view of (2.24), (2.25), [1] declares: this point cannot belong to the interval $0 < \eta < 1$ for $m \geqslant 1$. But this is not trivial, the essential point lies in

Problem 7. Why at the maximum point of $\tilde{\Phi}^{m,k}$, $(\tilde{\Phi}^{m,k} - \tilde{\Phi}^{m,k-1})/h \geqslant 0$? Actually only if $\tilde{\Phi}^{m,k-1} \leqslant 0$ or $\tilde{\Phi}^{m,k} - \tilde{\Phi}^{m,k-1} \geqslant 0$, then $(\tilde{\Phi}^{m,k} - \tilde{\Phi}^{m,k-1})/h \geqslant 0$. But generally, we cannot deduce that $\tilde{\Phi}^{m,k-1} \leqslant 0$ or $\tilde{\Phi}^{m,k} - \tilde{\Phi}^{m,k-1} \geqslant 0$.

However, this problem can also be solved by the method used to solve Problem 1.4, we omit details here. So applying the maximum principle, one has the conclusion that $\Phi^{m,k} \leqslant M_1$.

Now, consider the functions $X^{m,k} = \tilde{Y}^{m,k} - 2^{-1}M_2(1-\eta+h)^2$. It follows from (2.25) and (2.26) that

$$\begin{aligned}
\nu a^{m-1,k} X^{m,k}_{\eta\eta} &- e^{-\gamma h}\frac{X^{m,k} - X^{m-1,k}}{h} - \eta U^{m,k}\frac{X^{m,k} - X^{m,k-1}}{h} \\
&+ \tilde{A}^{m,k} X^{m,k}_{\eta} + (2\tilde{B}^{m,k} + Q^{m,k}_1 - \gamma e^{-\gamma h'})X^{m,k} \\
\geqslant & - K_{32}(M_1, M_2)(1-\eta+h)^2 \\
& - \frac{M_2}{2}[2a^{m-1,k} - 2\tilde{A}^{m,k}(1-\eta+h) \\
&\qquad + (2\tilde{B}^{m,k} + Q^{m,k}_1 - \gamma e^{-\gamma h'})(1-\eta+h)^2] \\
\geqslant & - K_{32}(M_1, M_2)(1-\eta+h)^2 \\
& - \frac{M_2}{2}[K_{34} - K_{35}](1-\eta+h)^2 \geqslant 0,
\end{aligned} \tag{2.27}$$

if $m < m'$ or $m = m'$, $k \leqslant k'$. Let us show that $X^{m,k} \leqslant 0$ for such m and k. If $X^{m,k}(\eta)$ assumes positive values, then there is a point η at which, for $m < m'$ or $m = m'$, $k \leqslant k'$, the function $X^{m,k}(\eta)$ attains its largest positive value. [1] declares: this point cannot belong to the interval $0 < \eta < 1$ for $m \geqslant 1$ because of (2.26). But this is not trivial, the essential point also lies in

Problem 8. Why at the maximum point of $X^{m,k}$, $(X^{m,k} - X^{m,k-1})/h \geqslant 0$? Actually only if $X^{m,k-1} \leqslant 0$ or $X^{m,k} - X^{m,k-1} \geqslant 0$, then $(X^{m,k} - X^{m,k-1})/h \geqslant 0$. But generally, we cannot deduce that $X^{m,k-1} \leqslant 0$ or $X^{m,k} - X^{m,k-1} \geqslant 0$.

However, this problem can also be solved by the method used to solve Problem 1.4, we omit details here, too.

After solving this problem, one is able to prove that $(1-\eta+h)w_{\eta\eta}^{m,k}$ are uniformly bounded in Ω. Lemma 5 is proved. $\square$

3. Oleinik's results

After the above modifications of proofs we can get the results of [1]. For the completeness of the paper, we quote here the last results which were obtained by Oleinik in [1].

Theorem 9. *Under the assumptions of Lemma* 1.3 *and Lemma* 2.1, *problem* (1.6)–(1.7) *in* Ω *with* $T = T_1$ *admits a solution* w *with the following properties:* w *is continuous in* Ω;

$$C_1(1-\eta) \leqslant w \leqslant C_2(1-\eta), \quad C_i = \text{const} > 0, \;\; i = 1,2; \tag{3.1}$$

w *has bounded weak derivatives* w_η, w_τ, w_ξ;

$$|w_\xi| \leqslant C_3(1-\eta), \quad |w_\tau| \leqslant C_4(1-\eta) \quad C_i = \text{const} > 0, \;\; i = 3,4; \tag{3.2}$$

the derivative w_η *is continuous in* $\eta < 1$; *conditions* (1.6) *hold for* w; *the weak derivative* $w_{\eta\eta}$ *exists and* $ww_{\eta\eta}$ *is bounded in* Ω; *equation* (1.5) *holds almost everywhere in* Ω. *The solution* w *of problem with these properties is unique.*

Theorem 10. *Assume that* U_x, U_t/U, Ur_x/r, v_0 *are bounded functions having bounded derivatives with respect to* $t, x \in D$; $u_0(x,y) \to U(0,x)$ *as* $y \to \infty$, $u_0 = 0$ *for* $y = 0$; u_0/U, u_{0y}/U *are continuous in* $\overline{D}$; $u_{0y} > 0$ *for* $y \geqslant 0$, $x > 0$,

$$K_1(U(0,x) - u_0(x,y)) \leqslant u_{0y}(x,y) \leqslant K_2(U(0,x) - u_0(x,y))$$

with positive constants K_1 *and* K_2. *Assume also that there exist bounded derivatives* u_{0y}, u_{0yy}, u_{0yyy}, u_{0x}, u_{0xy} *and the ratios*

$$\frac{u_{0yy}}{u_{0y}}, \quad \frac{u_{0yyy} - u_{0yy}^2}{u_{0y}^2}$$

are bounded for $0 \leqslant x \leqslant X$, $0 \leqslant y < \infty$. *Let the following compatibility condition be satisfied:*

$$v_0(0,x)u_{0y}(x,0) = -p_x(0,x) + \upsilon u_{0yy}(x,0), \tag{3.3}$$

and let

$$\left|\frac{u_{0yx} - u_{0x}u_{0yy}}{u_{0y}} + U_x\frac{u_0u_{0yy} - u_{0y}^2}{Uu_{0y}}\right| \leqslant K_5(U - u_0(x, y)).$$

Then problem (1.1)–(1.4) *in D has a unique solution u, v with the following properties: u/U, u_y/U are continuous and bounded in $\overline{D}$; $u_y/U > 0$ for $y \geqslant 0$; $u_y/U \to 0$ as $y \to \infty$; $u = 0$ for $y = 0$; v is continuous in y and bounded for bounded y; the weak derivatives u_t, u_x, u_{yt}, u_{yx}, u_{yy}, u_{yyy}, v_y are bounded measurable functions in D; the equations of system* (1.1) *hold almost everywhere in D; the functions u_t, u_x, v_y, u_{yy} are continuous with respect to y; moreover,*

$$\frac{u_{yy}}{u_y}, \quad \frac{u_{yyy} - u_{yy}^2}{u_y^2} \tag{3.4}$$

are bounded and the following inequalities hold:

$$C_1(U(t, x) - u(t, x, y)) \leqslant u_y(t, x, y) \leqslant C_2(U(t, x) - u(t, x, y)), \tag{3.5}$$

$$\exp(-C_2y) \leqslant 1 - \frac{u(t, x, y)}{U(t, x)} \leqslant \exp(-C_1y), \tag{3.6}$$

$$\left|\frac{u_{yt}u_y - u_tu_{yy}}{u_y} + U_t\frac{u_{yy}u - u_y^2}{u_yU}\right| \leqslant C_3(U - u), \tag{3.7}$$

$$\left|\frac{u_{yx}u_y - u_xu_{yy}}{u_y} + U_x\frac{u_{yy}u - u_y^2}{u_yU}\right| \leqslant C_4(U - u). \tag{3.8}$$

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